Extremum Seeking for Nash Games in Financial and Energy Markets

Miroslav Krstic
(joint work with Paul Frihauf and Tamer Basar)

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An Averaging Example

\[
\dot{x} = -\sin(\omega t) (x + \sin(\omega t))
\]

\[
\dot{x} \text{ ave} = -x^2 \sin(\omega t)
\]

\[
\dot{x} \text{ ave} = -\frac{1}{2} \sin^3(\omega t)
\]
An Averaging Example

\[ \dot{x} = -\sin(\omega t) (x + \sin(\omega t))^2 \]
An Averaging Example

\[ \dot{x} = - \sin(\omega t) (x + \sin(\omega t))^2 \]

\[ = -x^2 \sin(\omega t) \underbrace{- 2x \sin^2(\omega t) - \sin^3(\omega t)}_{\text{ave} = 0, \text{ave} = \frac{1}{2}, \text{ave} = 0} \]
An Averaging Example

\[ \dot{x} = -\sin(\omega t) (x + \sin(\omega t))^2 \]

\[ = -x^2 \underbrace{\sin(\omega t)}_{\text{ave} = 0} - 2x \underbrace{\sin^2(\omega t)}_{\text{ave} = \frac{1}{2}} - \sin^3(\omega t) \underbrace{_{\text{ave} = 0}}_{} \]

\[ \dot{x}_{\text{ave}} = -x_{\text{ave}} \]
Theorem 1 For sufficiently large $\omega$, there exists a \textit{locally exponentially stable periodic solution} $x^{2\pi/\omega}(t)$ such that

$$\left|x^{2\pi/\omega}(t)\right| \leq O\left(\frac{1}{\omega}\right) , \quad \forall t \geq 0.$$
Theorem 1  For sufficiently large $\omega$, there exists a locally exponentially stable periodic solution $x^{2\pi/\omega}(t)$ such that

$$
|x^{2\pi/\omega}(t)| \leq O\left(\frac{1}{\omega}\right), \quad \forall t \geq 0.
$$

Corollary 1  For sufficiently large $\omega$, there exist $M, m > 0$ such that

$$
|x(t)| \leq M|x(0)|e^{-mt} + O\left(\frac{1}{\omega}\right), \quad \forall t \geq 0.
$$
Extremum Seeking—the Basic Idea

\[ f^* + \frac{f''}{2}(\theta - \theta^*)^2 \]

\[ \hat{\theta} = \frac{k}{s} \]

\[ a \sin(\omega t) \]

\[ \text{sgnk} = -\text{sgn}f'' \]
Extremum Seeking—the Basic Idea

\[ f^* + \frac{f''}{2} (\theta - \theta^*)^2 \]

\[ k \]

\[ \frac{\text{d}^2 \tilde{\theta}}{\text{d}t^2} = ka \sin(\omega t) \left[ f^* + \frac{f''}{2} (\tilde{\theta} + a \sin(\omega t))^2 \right] \]
Theorem 2
There exists sufficiently large \( \omega \) such that, locally, 
\[
|\theta(t) - \theta^*| \leq |\theta(0) - \theta^*| e^{\int_0^t k f''(a^2 \tilde{\theta}_\text{ave}) dt} + O(\frac{1}{\omega}) + a,
\]
for all \( t \geq 0 \).
\[ \frac{d\tilde{\theta}_{\text{ave}}}{dt} = \frac{\kappa f'''}{k f'' a^2} \tilde{\theta}_{\text{ave}} \]

**Theorem 2**  *There exists sufficiently large \( \omega \) such that, locally,*

\[ |\theta(t) - \theta^*| \leq |\theta(0) - \theta^*| e^{\frac{k f''' a^2}{2} t} + O\left(\frac{1}{\omega}\right) + a, \quad \forall t \geq 0. \]
$f^*$ - unknown!

$f(\theta(t))$
Real-Time Optimization by Extremum-Seeking Control

KARTIK B. ARIYUR
MIROSLAV KRSTIĆ
Non-Cooperative Games

Multiple players, multiple cost functions.

Team optimization — ‘easy’ multivariable problems.

Selfish optimization — harder, because overall convexity is lost.
Non-Cooperative Games

Multiple players, multiple cost functions.

Team optimization — ‘easy’ multivariable problems.

Selfish optimization — harder, because overall convexity is lost.

Simplest case: two players, zero-sum game ($H_{\infty}$ control). One saddle surface, equilibrium at the saddle point.

Harder: two players, non-zero sum. Two saddle surfaces, equilibrium at the intersection of their “ridges.”

Even harder: 3 or more players, Nash game
Two Players — Duopoly

Coca-Cola vs. Pepsi
Boeing vs. Airbus
Two Players — Duopoly

Coca-Cola vs. Pepsi
Boeing vs. Airbus

Let $f_A$ and $f_B$ be two firms that produce the same good and compete for profit by setting their respective prices, $v_A$ and $v_B$.

Profit model:

\[
J_A(t) = i_A(t) (v_A(t) - m_A),
\]
\[
J_B(t) = i_B(t) (v_B(t) - m_B),
\]

where $i_A$ and $i_B$ are the number of sales and $m_A$ and $m_B$ are the marginal costs.
Two Players — Duopoly

Coca-Cola vs. Pepsi
Boeing vs. Airbus

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\]

where \( i_A \) and \( i_B \) are the number of sales and \( m_A \) and \( m_B \) are the marginal costs.

Sales model where the consumer prefers \( f_A \):

\[
i_A(t) = I - i_B(t), \quad i_B(t) = \frac{v_A(t) - v_B(t)}{p},
\]

where \( I \) are the total sales and \( p > 0 \) quantifies the preference of the consumer for \( f_A \).
The profit functions $J_A(v_A, v_B)$ and $J_B(v_A, v_B)$ are both quadratic functions of the prices $v_A$ and $v_B$.

The Nash strategies are

$$v_A^* = \frac{2m_A + m_B + 2Ip}{3}, \quad v_B^* = \frac{m_A + 2m_B + Ip}{3}.$$

How can the players ever know each other’s marginal costs, the customers’ preference, or the overall market demand?
Extremum seeking applied by firms $f_A$ and $f_B$ in a duopoly
Simulation with $m_A = m_B = 30$, $I = 100$, $p = 0.2$. 

![Prices graph](image-url)
Profits

$J_A(t)$

$J_B(t)$

time (sec)
Theorem 3  Let $\omega_A \neq \omega_B$, $2\omega_A \neq \omega_B$, and $\omega_A \neq 2\omega_B$. There exists $\omega^*$ such that, for all $\omega_A, \omega_B > \omega^*$, if $|\Delta(0)|$ is sufficiently small, then for all $t \geq 0$,

$$|\Delta(t)| \leq Me^{-mt}|\Delta(0)| + O\left(\frac{1}{\min(\omega_A, \omega_B)} + \max(a_A, a_B)\right),$$

where

$$\Delta(t) = \left( v_A(t) - v_A^*, \ v_B(t) - v_B^* \right)^T$$

$$M = \sqrt{\frac{\max(k_Aa_A^2, k_Ba_B^2)}{\min(k_Aa_A^2, k_Ba_B^2)}}$$

$$m = \frac{1}{2p}\min(k_Aa_A^2, k_Ba_B^2)$$
Theorem 3  Let $\omega_A \neq \omega_B$, $2\omega_A \neq \omega_B$, and $\omega_A \neq 2\omega_B$. There exists $\omega^*$ such that, for all $\omega_A, \omega_B > \omega^*$, if $|\Delta(0)|$ is sufficiently small, then for all $t \geq 0$,

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where

$$\Delta(t) = (v_A(t) - v_A^*, v_B(t) - v_B^*)^T$$

$$M = \sqrt{\frac{\max(k_Aa_A^2, k_Ba_B^2)}{\min(k_Aa_A^2, k_Ba_B^2)}}$$

$$m = \frac{1}{2p}\min(k_Aa_A^2, k_Ba_B^2)$$

Proof. Let $\tau = \omega t$ and $\omega = \min(\omega_A, \omega_B)$. The average system is

$$\frac{d}{d\tau} \begin{pmatrix} \hat{v}_A^\text{ave} \\ \hat{v}_B^\text{ave} \end{pmatrix} = \frac{1}{2\omega p} \begin{pmatrix} -2k_Aa_A^2 & k_Aa_A^2 \\ k_Ba_B^2 & -2k_Ba_B^2 \end{pmatrix} \begin{pmatrix} \hat{v}_A^\text{ave} \\ \hat{v}_B^\text{ave} \end{pmatrix}.$$
Oligopoly ($N$ Players)

Let $m_i$ be the marginal cost of $f_i$, $i_i$ its sales volume, and the profit is

$$J_i(t) = i_i(t)(v_i(t) - m_i)$$

A model of sales $i_1$, $i_2$, $i_3$ in a three-firm oligopoly with prices $v_1$, $v_2$, $v_3$, and total sales $I$. The desirability of product $i$ is proportional to $1/R_i$. 


For \( N \) players, the sales volume is obtained as

\[
i_i(t) = \frac{R_{||}}{R_i} \left( I - v_i(t) \frac{1}{\bar{R}_i} + \sum_{j=1}^{N} \frac{v_j(t)}{R_j} \right),
\]

\[
R_{||} = \left( \sum_{k=1}^{N} \frac{1}{R_k} \right)^{-1}, \quad \bar{R}_i = \left( \sum_{k=1}^{N} \frac{1}{R_k} \right)_{k \neq i}^{-1}
\]
The Nash equilibrium of this game is given by

\[ v_i^* = \frac{\Lambda R_i}{2R_i + \bar{R}_i} \left( \bar{R}_i I + m_i + \sum_{j=1}^{N} \frac{m_j \bar{R}_i - m_i \bar{R}_j}{2R_j + \bar{R}_j} \right), \]

where

\[ \Lambda = \left( 1 - \sum_{j=1}^{N} \frac{\bar{R}_j}{2R_j + \bar{R}_j} \right)^{-1} > 0 \]
\[ J_i(t) = i_i(t)(v_i(t) - m_i) \]

The profit \( J_i \), in electrical analogy, corresponds to the power absorbed by the \( v_i - m_i \) portion of the voltage generator \( i \).
Extremum seeking strategy:

\[
\frac{d\hat{v}_i(t)}{dt} = k_i \mu_i(t) J_i(t)
\]
\[
\mu_i(t) = a_i \sin(\omega_i t + \varphi_i)
\]
\[
v_i(t) = \hat{v}_i(t) + \mu_i(t)
\]
Simulation with $m_1 = 22$, $m_2 = 20$, $m_3 = 26$, $m_4 = 20$, $I = 100$, $R_1 = 0.25$, $R_2 = 0.78$, $R_3 = 1.10$, and $R_4 = 0.40$. 

![Graph showing prices over time with four lines for $v_1(t)$, $v_2(t)$, $v_3(t)$, and $v_4(t)$ with time in seconds ranging from 0 to 200.]
The diagram illustrates the profits over time, with four distinct functions:

- $J_1(t)$
- $J_2(t)$
- $J_3(t)$
- $J_4(t)$

The x-axis represents time in seconds, while the y-axis represents profits. Each function is plotted with different colors, indicating distinct performance trends.
Theorem 4  \textit{Let } ω_i \neq ω_j, \ 2ω_i \neq ω_j \text{ for all } i \neq j, \ i, j = 1, \ldots, N. \text{ There exists } ω^* \text{ such that, for all } \min_i ω_i > ω^*, \text{ if } |Δ(0)| \text{ is sufficiently small, then for all } t \geq 0,}

\[ |Δ(t)| \leq \Xi e^{-ξt} |Δ(0)| + O \left( \frac{1}{\min_i ω_i} + \max_i a_i \right), \]

where

\[ Δ(t) = (v_1(t) - v_1^*, \ldots, v_N(t) - v_N^*)^T \]

\[ \Xi = \sqrt{\max_i \{k_i a_i^2\}} \]

\[ ξ = \frac{R \|\min_i \{k_i a_i^2\}\|}{2 \max_i \{R_i Γ_i\}} \]

\[ Γ_i = \min_{\substack{j \in \{1, \ldots, N\}, j \neq i}} R_j \]
Proof. Let $\tau = \omega t$ where $\omega = \min_i \omega_i$. The average system is obtained as $\frac{d}{d\tau} \tilde{v}_{\text{ave}} = A \tilde{v}_{\text{ave}}$ where

$$A = \frac{R_{||}}{2\omega} \begin{pmatrix} -\frac{2k_1 a_1^2}{R_1 R_1} & \frac{k_1 a_1^2}{R_1 R_2} & \cdots & \frac{k_1 a_1^2}{R_1 R_N} \\ \frac{k_2 a_2^2}{R_2 R_1} & -\frac{2k_2 a_2^2}{R_2 R_2} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{k_N a_N^2}{R_N R_1} & \cdots & \cdots & -\frac{2k_N a_N^2}{R_N R_N} \end{pmatrix}.$$ 

Let $V = (\tilde{v}_{\text{ave}})^T P \tilde{v}_{\text{ave}}$ be a Lyapunov function, where $P = \frac{\omega}{R_{||}} \text{diag} \left( \frac{1}{k_1 a_1^2}, \ldots, \frac{1}{k_N a_N^2} \right)$ and satisfies the Lyapunov equation $PA + A^T P = -Q,$

$$Q = \begin{pmatrix} \frac{2}{R_1 R_1} & -\frac{1}{R_1 R_2} & \cdots & -\frac{1}{R_1 R_N} \\ -\frac{1}{R_2 R_1} & \frac{2}{R_2 R_2} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{R_N R_1} & \cdots & \cdots & \frac{2}{R_N R_N} \end{pmatrix}.$$
The matrix $Q$ is positive definite symmetric and diagonally dominant, namely,

$$\sum_{j=1}^{N} |q_{i,j}| = \frac{1}{R_i \bar{R}_i} < \frac{2}{R_i \bar{R}_i} = |q_{i,i}|.$$  

From the Gershgorin Theorem, $\lambda_i(Q) \in \frac{1}{R_i \bar{R}_i} [1, 3]$, which implies that

$$\lambda_{\min}(Q) > \frac{1}{\max_i \{R_i \bar{R}_i\}} > \frac{1}{\max \{R_i \Gamma_i\}}.$$  

Q.E.D.
Continuum of Players

Traditional Energy Production

Power Grid

Renewable Energy Production

Households
Oligopoly w/ uncountably many non-atomic players, indexed by continuum index $x \in [0, 1]$.

The profit of firm $f(x)$:

$$J(x,t) = i(x,t) (v(x,t) - m(x)),$$

with the sales modeled as

$$i(x,t) = \frac{R||}{R(x)} \left( I - \frac{v(x,t)}{R||} + \int_0^1 \frac{v(y,t)}{R(y)} \, dy \right),$$

$$R|| = \left( \int_0^1 \frac{dy}{R(y)} \right)^{-1}.$$
Oligopoly w/ uncountably many non-atomic players, indexed by continuum index \( x \in [0, 1] \).

The profit of firm \( f(x) \):

\[
J(x,t) = i(x,t) \left( v(x,t) - m(x) \right),
\]

with the sales modeled as

\[
i(x,t) = \frac{R_{\|}}{R(x)} \left( I - \frac{v(x,t)}{R_{\|}} + \int_0^1 \frac{v(y,t)}{R(y)} dy \right),
\]

\[
R_{\|} = \left( \int_0^1 \frac{dy}{R(y)} \right)^{-1}.
\]

The Nash equilibrium values of the prices and the corresponding sales are

\[
v^*(x) = R_{\|} \left( I + \frac{1}{2} \frac{m(x)}{R_{\|}} + \frac{1}{2} \int_0^1 \frac{m(y)}{R(y)} dy \right),
\]
\[
i^*(x) = \frac{R_{\|}}{R(x)} \left( I - \frac{1}{2} \frac{m(x)}{R_{\|}} + \frac{1}{2} \int_0^1 \frac{m(y)}{R(y)} dy \right).
\]
Extremum seeking algorithm:

\[
\frac{\partial}{\partial t} \hat{v}(x, t) = k(x)\mu(x, t)J(x, t)
\]

\[
\mu(x, t) = a(x) \sin(\omega(x) t + \varphi(x))
\]

\[
v(x, t) = \hat{v}(x, t) + \mu(x, t)
\]

where \(a(x), k(x) > 0\), for all \(x \in [0, 1]\).
Extremum seeking algorithm:
\[
\frac{\partial}{\partial t} \hat{v}(x,t) = k(x) \mu(x,t) J(x,t)
\]
\[
\mu(x,t) = a(x) \sin(\omega(x)t + \phi(x))
\]
\[
v(x,t) = \hat{v}(x,t) + \mu(x,t)
\]
where \(a(x), k(x) > 0\), for all \(x \in [0, 1]\).

Let \(\Omega_\omega\) be the set of bounded functions \(\omega : [0, 1] \rightarrow \mathbb{R}_+\) such that, at each element of the set \(\omega([0, 1]) \cup 2\omega([0, 1])\), the level set of \(\omega\) is of measure zero, and where \(\min_{x \in [0,1]} \omega(x) > 0\).
Extremum seeking algorithm:

\[
\frac{\partial}{\partial t} \hat{v}(x,t) = k(x)\mu(x,t)J(x,t)
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\[
\mu(x,t) = a(x)\sin(\omega(x)t + \varphi(x))
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Let \(\Omega_\omega\) be the set of bounded functions \(\omega : [0,1] \to \mathbb{R}_+\) such that, at each element of the set \(\omega([0,1]) \cup 2\omega([0,1])\), the level set of \(\omega\) is of measure zero, and where \(\min_{x \in [0,1]} \omega(x) > 0\).

(The set \(\Omega_\omega\) contains all functions that are either strictly increasing or strictly decreasing, as well as all bounded \(C^1[0,1]\) positive functions whose derivative is zero on a set of measure zero.)
Theorem 5  There exists $\omega^*$ such that, for all functions $\omega \in \Omega_{\omega^*}$, if the $L_2[0, 1]$ norm of $\Delta(x, 0)$ is sufficiently small, then for all $t \geq 0$,

$$
\int_0^1 \Delta^2(x, t) dx \leq \Sigma e^{-\sigma t} \int_0^1 \Delta^2(x, 0) dx + O \left( \frac{1}{\min_x \omega^2(x)} + \max_x a^2(x) \right),
$$

where

$$
\Delta(x, t) = v(x, t) - v^*(x)
$$

$$
\Sigma = \frac{\max_x \{k(x)a^2(x)\}}{\min_x \{k(x)a^2(x)\}}
$$

$$
\sigma = \frac{\min_x \{k(x)a^2(x)\}}{\max_x \{R(x)\}}
$$
Proof. Error system

$$\frac{\partial}{\partial t} \tilde{v}(x,t) = \frac{k(x)}{R(x)} G[\tilde{v}, R, i^*, \mu](x,t),$$

with the operator $G$ defined as

$$G[\tilde{v}, R, i^*, \mu](x,t) \equiv \mu(x,t) \left[ \left( R(x)i^*(x) - \tilde{v}(x,t) + \left\langle \frac{R}{R}, \tilde{v} \right\rangle \right) \left( R(x)i^*(x) + \tilde{v}(x,t) \right) 
+ \mu(x,t) \left( -2\tilde{v}(x,t) + \left\langle \frac{R}{R}, \tilde{v} \right\rangle \right) 
+ \left\langle \frac{R}{R}, \mu \right\rangle \left( R(x)i^*(x) + \tilde{v}(x,t) \right) + \mu(x,t) \left\langle \frac{R}{R}, \mu \right\rangle - \mu^2(x,t) \right],$$

where $\left\langle a, b \right\rangle \equiv \int_0^1 a(y)b(y)dy$.

Recall: $\mu(x,t) = a(x) \sin(\omega(x)t + \varphi(x))$
Let $\omega = \min_x \{ \omega(x) \}$, $\gamma(x) = \omega(x)/\omega$, and $\tau = \omega t$.

To apply infinite-time averaging ("general averaging") to the infinite dimensional system, we have to compute various integrals and verify the conditions of the dominated convergence theorem for their integrands, to justify swapping the order of integrals in $x$ and limits in $\tau$. 
Let $\omega = \min_x \{ \omega(x) \}$, $\gamma(x) = \omega(x)/\omega$, and $\tau = \omega t$.

To apply infinite-time averaging (“general averaging”) to the infinite dimensional system, we have to compute various integrals and verify the conditions of the dominated convergence theorem for their integrands, to justify swapping the order of integrals in $x$ and limits in $\tau$.

We obtain the average system

$$\frac{\partial}{\partial \tau} \bar{\nu}^{\text{ave}}(x, \tau) = -\frac{k(x)a^2(x)}{\omega R(x)} \bar{\nu}^{\text{ave}}(x, \tau) + \frac{R||k(x)a^2(x)}{2 \omega R(x)} \int_0^1 \frac{\bar{\nu}^{\text{ave}}(y, \tau)}{R(y)} dy$$
Let $V(\tau)$ be a Lyapunov functional defined as

$$V(\tau) = \frac{\omega}{2} \int_0^1 \frac{1}{k(x)a^2(x)} (\tilde{v}^\text{ave})^2 (x, \tau) \, dx$$

and bounded by

$$\frac{\omega \int_0^1 (\tilde{v}^\text{ave})^2 (x, \tau) \, dx}{2 \max_x \{k(x)a^2(x)\}} \leq V(\tau) \leq \frac{\omega \int_0^1 (\tilde{v}^\text{ave})^2 (x, \tau) \, dx}{2 \min_x \{k(x)a^2(x)\}}$$

Taking the time derivative and applying the Cauchy-Schwarz inequality, we obtain

$$\dot{V} \leq -\frac{1}{2} \int_0^1 \frac{(\tilde{v}^\text{ave})^2 (x, \tau)}{R(x)} \, dx$$

From the infinite-dimensional averaging theory in [Hale and Verduyn Lunel, 1990], the result of the theorem follows.

Q.E.D.
General *Nonquadratic* Games with \( N \) Players

Consider the payoff function of player \( i \):

\[
J_i = h_i(u_i, u_{-i})
\]

where \( u_i \in \mathbb{R} \) is player \( i \)'s action and \( u_{-i} = [u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N] \) represents the actions of the other players.
**General Nonquadratic Games with $N$ Players**

Consider the payoff function of player $i$:

$$J_i = h_i(u_i, u_{-i})$$

where $u_i \in \mathbb{R}$ is player $i$'s action and $u_{-i} = [u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N]$ represents the actions of the other players.

**ES strategy:**

$$\hat{u}_i(t) = k_i\mu_i(t)J_i(t)$$

$$\mu_i(t) = a_i \sin(\omega_i t + \varphi_i)$$

$$u_i(t) = \hat{u}_i(t) + \mu_i(t)$$
**Assumption 1** There exists at least one (possibly multiple) isolated Nash equilibrium \( u^* = [u_1^*, \ldots, u_N^*] \) such that

\[
\frac{\partial h_i}{\partial u_i}(u^*) = 0, \quad \frac{\partial^2 h_i}{\partial u_i^2}(u^*) < 0,
\]

for all \( i \in \{1, \ldots, N\} \).
Assumption 1  There exists at least one (possibly multiple) isolated Nash equilibrium $u^* = [u_1^*, \ldots, u_N^*]$ such that

$$\frac{\partial h_i}{\partial u_i}(u^*) = 0, \quad \frac{\partial^2 h_i}{\partial u_i^2}(u^*) < 0,$$

for all $i \in \{1, \ldots, N\}$.

Assumption 2  The matrix

$$\Lambda = \begin{bmatrix}
\frac{\partial^2 h_1(u^*)}{\partial u_1^2} & \frac{\partial^2 h_1(u^*)}{\partial u_1\partial u_2} & \cdots & \frac{\partial^2 h_1(u^*)}{\partial u_1\partial u_N} \\
\frac{\partial^2 h_2(u^*)}{\partial u_1\partial u_2} & \frac{\partial^2 h_2(u^*)}{\partial u_2^2} & \cdots & \frac{\partial^2 h_2(u^*)}{\partial u_1\partial u_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 h_N(u^*)}{\partial u_1\partial u_N} & \frac{\partial^2 h_N(u^*)}{\partial u_2\partial u_N} & \cdots & \frac{\partial^2 h_N(u^*)}{\partial u_N^2}
\end{bmatrix}$$

is diagonally dominant and hence, nonsingular.
Theorem 6  Let $\omega_i \neq \omega_j$, $\omega_i \neq \omega_j + \omega_k$, $2\omega_i \neq \omega_j + \omega_k$, and $\omega_i \neq 2\omega_j + \omega_k$ for all $i$, $j$, $k \in \{1, \ldots, N\}$. Then there exists $\omega^*$, $\bar{a}$ and $M$, $m > 0$ such that, for all $\min_i \omega_i > \omega^*$ and $\alpha_i \in (0, \bar{a})$, if $|\Delta(0)|$ is sufficiently small, then for all $t \geq 0$,

$$|\Delta(t)| \leq Me^{-mt} |\Delta(0)| + O \left( \max_i a_i^3 \right),$$

where

$$\Delta(t) = \left[ \hat{u}_1(t) - u_1^* - \sum_{j=1}^{N} c_{jj}^1 \alpha_j^2, \ldots, \hat{u}_N(t) - u_N^* - \sum_{j=1}^{N} c_{jj}^N \alpha_j^2 \right]$$

$$\begin{bmatrix}
  c_{ii}^1 \\
  \vdots \\
  c_{ii}^{i-1} \\
  c_{ii}^i \\
  c_{ii}^{i+1} \\
  \vdots \\
  c_{ii}^N
\end{bmatrix} = -\frac{1}{4} \Lambda^{-1}
$$

$$\begin{bmatrix}
  \frac{\partial^3 h_1}{\partial u_1 \partial u_i^2} (u^*) \\
  \vdots \\
  \frac{\partial^3 h_{i-1}}{\partial u_{i-1} \partial u_i^2} (u^*) \\
  \frac{1}{2} \frac{\partial^3 h_i}{\partial u_i^3} (u^*) \\
  \frac{\partial^3 h_{i+1}}{\partial u_i^2 \partial u_{i+1}} (u^*) \\
  \vdots \\
  \frac{\partial^3 h_N}{\partial u_i^2 \partial u_N} (u^*)
\end{bmatrix} $$
Numerical Example with Dynamics and Non-Quadratic Payoffs

\[ \begin{align*}
    a_1 \sin(\omega_1 t + \varphi_1) & \quad \hat{u}_1 \\
    \frac{k_1}{s} & \\
    \hat{u}_2 & \\
    a_2 \sin(\omega_2 t + \varphi_2) &
\end{align*} \]
\[ \dot{x}_1 = -4x_1 + x_1x_2 + u_1 \]
\[ \dot{x}_2 = -4x_2 + u_2 \]
\[ J_1 = -16x_1^2 + 8x_1^2x_2 - x_1^2x_2^2 - 4x_1x_2^2 + 15x_1x_2 + 4x_1 \]
\[ J_2 = -64x_2^3 + 48x_1x_2 - 12x_1x_2^2 \]
\[
\dot{x}_1 = -4x_1 + x_1 x_2 + u_1 \\
\dot{x}_2 = -4x_2 + u_2 \\
J_1 = -16x_1^2 + 8x_1^2 x_2 - x_1^2 x_2^2 - 4x_1 x_2^2 + 15x_1 x_2 + 4x_1 \\
J_2 = -64x_2^3 + 48x_1 x_2 - 12x_1 x_2^2
\]

**Steady-state payoffs**

\[
J_1 = -u_1^2 + u_1 u_2 + u_1 \\
J_2 = -u_2^3 + 3u_1 u_2
\]
\[
\begin{align*}
\dot{x}_1 &= -4x_1 + x_1x_2 + u_1 \\
\dot{x}_2 &= -4x_2 + u_2 \\
J_1 &= -16x_1^2 + 8x_1^2x_2 - x_1^2x_2^2 - 4x_1x_2^2 + 15x_1x_2 + 4x_1 \\
J_2 &= -64x_2^3 + 48x_1x_2 - 12x_1x_2^2
\end{align*}
\]

Steady-state payoffs

\[
\begin{align*}
J_1 &= -u_1^2 + u_1u_2 + u_1 \\
J_2 &= -u_2^3 + 3u_1u_2
\end{align*}
\]

Reaction curves

\[
\begin{align*}
l_1 &\triangleq \left\{ u_1 = \frac{1}{2} (u_2 + 1) \right\} \\
l_2 &\triangleq \left\{ u_2^2 = u_1 \right\}
\end{align*}
\]
\[ (u_1^* + \delta, u_2^* + 2\delta) \approx (u_1^* - a_2^2/12, u_2^* - a_2^2/6) \]

\[ (v_1^* + \delta, v_2^* + 2\delta) \approx (v_1^* + a_2^2/12, v_2^* + a_2^2/6) \]
Summary

- ES solves games w/o knowledge of the player’s own payoff model and w/o knowledge of the actions or payoffs of the opponents.

- Stochastic counterparts of the ES methods also developed (for finitely many players so far).